Phys 410
Fall 2013
Lecture #10 Summary
3 October, 2013

Making Newton's second law work in a rotating reference frame is a challenge. Consider a rigid body undergoing pure rotational motion on an axis through a fixed point inside the object. We found that the linear velocity of a particle at location $\vec{r}$ inside or on the object is given by $\vec{v} = \vec{\omega} \times \vec{r}$. In other words $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$, or in general for any vector $\vec{e}$ in the rigid body $\frac{d\vec{e}}{dt} = \vec{\omega} \times \vec{e}$.

We then calculated the relationship between the time-derivative of a vector $\vec{Q}$ as seen in an inertial reference frame S_0, to the derivative of the same vector seen in the rotating reference frame S. We assume that the two reference frames have the same origin, but frame S is rotating about an arbitrary axis $\hat{\Omega}$ through the origin at a rate Ω. The time-derivatives are related as $\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega} \times \vec{Q}$. This equation says that the time derivative of the vector as witnessed in the inertial reference frame consists of any change in its magnitude or direction as seen in the non-inertial reference frame, plus the change brought about by the fact that the vector $\vec{Q}$ is embedded in a rotating rigid body.

Newton's second law can now be written for an observer in a rotating reference frame as $m\ddot{\vec{r}} = \vec{F}_{net} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$. The two "inertial forces" on the right are called the Coriolis force and the centrifugal force, respectively.

We considered the centrifugal ("center-fleeing") force for a stationary observer on the surface of the earth. This force has a direction that is directly away from the axis of rotation of the earth and can be written as $\vec{\mathrm{F}}_{CF} = m\Omega^2 r \sin\theta\, \hat{\rho}$, where r is the distance from the center of the earth, θ is the polar angle of the location on the surface (also known as the co-latitude) and $\hat{\rho}$ is the radial unit vector from cylindrical coordinates. This force has a maximum magnitude near the equator, but goes to zero at the poles. The centrifugal force modifies the free-fall direction. It creates a new effective gravitational acceleration vector of $\vec{g} = \vec{g}_0 + \Omega^2 R \sin\theta\, \hat{\rho}$, where $\vec{g}_0$ is the bare Newtonian gravity acceleration vector that points directly to the center of the earth, and R is the radius of the earth. The radial component of this vector is $g_{rad} = g_0 - \Omega^2 R \sin^2\theta$, showing that things weigh a bit less at the equator than at the north/south pole. The effect is small, only about 0.3%. The tangential component of $\vec{g}$ is $g_{tang} = \Omega^2 R \sin\theta \cos\theta$, with a maximum value at 45° latitude. This component produces a 0.1° tilt of $\vec{g}$ with respect to the direction of $\vec{g}_0$.

The Coriolis force $\vec{F}_{Cor} = 2m\dot{\vec{r}} \times \vec{\Omega}$ depends on the state of motion of the object. In fact it resembles the force on a charged particle in a magnetic field. The 'charge' is $2m$ and the 'magnetic field' is the angular velocity vector $\vec{\Omega}$. The particle will be deflected as it travels through this 'field'. In the northern hemisphere the deflection is to the right, while in the southern hemisphere it is in the opposite direction because $\vec{\Omega}$ has a substantial component into the ground (hence the phrase 'down under'). The magnitude of the Coriolis force for an object on the surface of the earth moving at 50 m/s is quite small, resulting in an acceleration of at most 0.007 m/s^2. The Coriolis force is significant for objects with large mass (air masses, hurricanes, etc.), or for objects moving quickly (artillery shells and ICBMs).